Lecture notes: non-linearity and noise

Objective
The objective of these brief notes is to supplement the textbooks used in the course on the topic of non-linearity and electrical noise.

Non-linearity
Linearity in electronic circuits are often of paramount importance. Small non-linearities can cause undesired distortion of signals or even cause system malfunction. Audio equipment, for instance, need to be reasonably linear, or unpleasant distortion is heard. Scientific measurements equipment where very accurate measurements are needed need to be linear. Radio-transceivers, particularly the ones used in mobile phones, need to be extraordinarily linear, or channel interference occur. Figure 1 shows examples of distorted sinusoidal signals that are often encountered: clipping of the signal due to limited power supplies, cross-over distortion often seen in some classes of power amplifiers, and a slew-rate limited signal caused by the finite speed of the electronic circuit.

![Figure 1: Typical distortions of sinusoidal waveform: undistorted (a), clipped (b), cross-over (c), slew-rate (d), typical frequency spectrum (e).](image)

Non-linearity measures
The are several measures of non-linearity used in various electronics disciplines. In RF electronics, for instance, the “third-order inter-modulation intercept point” or the “1dB compression point” is often quoted. Which measure is used is often dictated by what is easily measurable. The non-linearity measure we adopt in this course is the popular Total Harmonic Distortion (THD). For this we need the frequency spectrum of a periodic waveform which is easily obtained at low frequencies using the FFT function on a digital CRO. To calculate the THD, we need the magnitude of the first harmonic (or fundamental), $V_1$ and also the magnitudes of all higher harmonics $V_2, V_3, V_4, \ldots$. The THD is then calculated thus (typically quoted in percent):

$$\text{THD} = \sqrt{\frac{V_2^2 + V_3^2 + V_4^2 + \cdots}{V_1^2}}.$$
Noise

All circuit elements that exhibit loss generate electrical noise. Ideal reactive components such as inductors and capacitors are noiseless, whereas resistors, transistors and diodes are noisy. There are several different noise mechanisms; some well understood, and some still a bit of a mystery to science. Common to most noise processes encountered in electronics is that it is random movements of electrons that generate the noise; thermal noise, for instance is caused by the random movements of electrons due to their thermal energy and increase proportionally with the absolute temperature. Figure 2 shows a typical noisy signal in the time domain and frequency domain; you can observe such a signal on your CRO by just turning the gain setting up high.

While distortion limits how large signals an electronic system can process, noise limits how small signals the system can process. The dynamic range of is the ratio between the largest and smallest signal that the system can process. Finite dynamic range is what ultimately limits the coverage of mobile phone networks, for instance.

![Electronic noise. Time domain signal (a), frequency domain signal (b).](image)

**Noise signals and noise in resistors**

Noise signals are (zero mean) random signals. They are described by their variance estimates or frequency spectres rather than their time-domain signals. The noise in resistors is usually modelled as a series noise voltage source \( v_N(t) \) as shown in Figure 3(a). The variance, \( V_N^2 \) of \( v_N(t) \) can be estimates thus:

\[
V_N^2 = \frac{1}{T} \int_0^T v_N^2(t) \, dt.
\]

The variance is often denoted the noise power\(^1\). Note that noise voltage sources are not normally shown with a polarity, as the polarity is always unimportant. Noise in different components are uncorrelated. Thus, to find the total noise power of the series connection of the two resistors in Figure 3(b), we calculate:

\[
V_{N_{tot}}^2 = \frac{1}{T} \int_0^T (v_{N1}(t) + v_{N2}(t))^2 \, dt = V_{N1}^2 + V_{N2}^2 + 2 \frac{1}{T} \int_0^T v_{N1}(t)v_{N2}(t) \, dt = V_{N1}^2 + V_{N2}^2
\]

where the co-variance term \( \int v_{N1}v_{N2} \, dt \) evaluates to zero for uncorrelated signals. Note here, that when finding the total noise one should thus always add the noise powers of all noise sources!\(^2\)

\(^1\)This notation is derived from signal processing where signals do not have units. You can think of the noise power as the power that would be dissipated in a 1Ω resistor.

\(^2\)While the instantaneous noise voltages obviously add, the interesting measure is the power of the signal, not the instantaneous value.
Noise signals are better described in the frequency domain than in the time domain. One can think of the frequency domain signal described as:

$$V_n^2(f) = \mathcal{F}(v_n^2(t)) = \int_{-\infty}^{\infty} e^{-j\omega t} v_n^2(t) dt,$$

and we thus find the noise power by integration in the frequency domain:

$$V_n^2 = \int_{0}^{\infty} V_n^2(f) df,$$

where $V_n^2(f)$ is the noise power spectral density (in $V^2$/Hz; datasheets often quote the noise spectral density $\sqrt{V_n^2(f)}$ in $V/\sqrt{Hz}$).³

All resistors are subject to thermal noise. The thermal noise power spectral density is independent of frequency (white noise), as shown in Figure 3(c), and have a value of:

$$V_n^2(f) = 4kTR \text{ (resistor)},$$

where $k$ is Boltmann’s constant, $T$ is the absolute temperature and $R$ is the value of the resistor.

![Figure 3: Noise model for resistor. Single resistor with series noise voltage (a), series connections of resistors (b), frequency spectrum of thermal noise in resistors (c).](image)

**Shaped noise**

To find the total noise in a circuit, you find the noise power from each noise source at the circuit output and add them all up. To illustrate this, let’s look at the simple circuit in Figure 4(a) and find the noise power across the capacitor. The transfer function from the noise source $V_n(f)$ to the capacitor voltage $V_c(f)$ (in the frequency domain) is $H(j\omega)$, and we find using normal circuit analysis:

$$V_c(f) = H(j\omega)V_n(f) = V_n(f) \cdot \frac{1}{1 + j\omega RC}.$$

Thus, we find the total noise power at the capacitor as:

$$V_C^2 = \int_{0}^{\infty} V_c^2(f) df = \int_{0}^{\infty} |H(j\omega)|^2 V_n^2(f) df = \int_{0}^{\infty} \frac{4kTR}{1 - (\omega RC)^2} df = 4kTR \cdot \frac{\pi/2}{2\pi RC} = \frac{kT}{C}.$$

Interestingly, even though the capacitor is a noiseless component, the total noise power depends on the capacitor only, and not on the resistor! We can generalise the integral to finding the

³The math is not strictly correct here; however $V_n^2(f)$ is usually given as a single-sided spectrum such that the integral should be taken over the frequencies 0 to $\infty$ as stated.
total output noise power in a circuit with \( N \) noise sources \( V_n(f) \) each having individual transfer functions to the output \( H_i(j\omega) \):

\[
V_{N_{\text{tot}}}^2 = \sum_i \int_0^\infty |H_i(j\omega)|^2 V_{ni}^2(f) \, df .
\]

Note that in the common special case where the noise source is white (\( V_n^2(f) = V_{nW}^2 \)), and the transfer function \( H(j\omega) \) is a single-pole lowpass filter (i.e. in the capacitor example above), one defines the noise bandwidth, \( B_N \) as

\[
B_N = \frac{\pi}{2} f_{3\text{dB}} ,
\]

where \( f_{3\text{dB}} \) is the filter 3 dB cut-off frequency, and one can simply calculate:

\[
V_N^2 = V_{nW}^2 B_N .
\]

Figure 4: Noise on capacitor connected to resistor. Schematic (a), frequency domain signal (b).

**Shot noise and noise in diodes**

Diodes and other devices (e.g. vacuum tubes) where electrons cross an energy boundary are subject to shot noise. Shot noise is white and is modelled as a parallel noise current source as shown in Figure 5. The noise power spectral density is:

\[
I_n^2(f) = 2qI_D \quad \text{(diode)},
\]

where \( q \) is the electron charge and \( I_D \) is the device current.

Figure 5: Shot noise current in diode. Model (a), frequency domain signal (b).
Pink noise and noise in MOSFETs

The resistive channel in a MOSFET is subject to thermal noise. Traditionally, this is modelled as
a noise current source in parallel with the drain-source path. The spectral density of this is:

\[ I_n^2(f) = 4kTg_m \frac{2}{3} \text{(MOSFET)}, \]

where \( g_m \) is the transistor transconductance. MOSFETs are also subject to pink noise (also known
as 1/f noise) which is associated with charge traps in the channel region. This is modelled as a
noise voltage source in series with the gate terminal:

\[ V_n^2(f) = \frac{K}{f} \text{(MOSFET)}, \]

where \( K \) is a suitable constant that depends on device area and other fabrication parameters.

![Diagram of noise in MOSFET](image)

Figure 6: Noise in MOSFET. Model (a), frequency domain signal of channel noise current (b),
frequency domain signal of gate noise voltage (c).

Signal to noise ratios

If signals are too small, they will be swamped by the inherent electrical noise in any electrical
system. The signal-to-noise ratio is a measure of how well clear of the noise a signal is. With a
signal-to-noise ratio of 100, say, there is usually little point trying digitise that signal with more
than about seven bits (128 levels) as the least significant bits will just sample the noise and be
random\(^4\). To find the signal-to-noise ratio (SNR), we need the total noise power \( V_N^2 \) found above

![Diagram of signal in noise](image)

Figure 7: Signal in noise. Voltage source with output resistance \( R \) and parasitic load capacitance
\( C \) (a), time domain output voltage (b), frequency domain output voltage (c).

\(^4\)Sometimes is is possible to do post-processing that can average out some of the noise, in which case it makes
sense to use a few extra bits in the digitisation. Digital CROs, for instance, normally have this capability.
as well the signal power (i.e. the square of the RMS value of the signal, $V_{\text{rms}}^2$); we then find:

$$\text{SNR} = \frac{V_{\text{rms}}^2}{V_N^2},$$

often quoted in dB (i.e. $10\log\left(\frac{V_{\text{rms}}^2}{V_N^2}\right)$). Figure 7 shows an example of a signal source with finite output resistance and capacitance, and time and frequency domain output signals.

**Dynamic range**

The dynamic range (DR) of a system is the ratio between the largest and the smallest signal that the system can process. In digital systems, for instance, the dynamic range relate to the number of bits used to represent signals ($\text{DR} = 2^{N_B}$, where $N_B$ is the number of bits). In analogue systems (or subsystems), the largest signal is limited by distortion while the smallest signal is limited by noise. The exact definition of what the largest and smallest signals are varies with applications and traditions in different disciplines. One reasonable measure is:

$$\text{DN} = \frac{V_{\text{rms, max}}^2}{V_N^2},$$

where $V_{\text{rms, max}}$ is the largest undistorted signal — this assumes that distortion sets in suddenly (for instance due to clipping of the signal) rather than steadily increasing with the signal amplitude, and that an SNR=1 is an acceptable minimum signal.