Introduction to Kalman Filter

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1 Introduction

First one must ask, what is a Kalman filter? A Kalman filter is simply an *optimal recursive data processing algorithm*. There are many ways of defining optimal, dependent upon the criteria chosen to evaluate performance. The word *recursive* in the previous description means that, unlike certain data processing concepts, the Kalman filter does not require all previous data to be kept in storage and reprocessed every time a new measurement is taken. The "filter" is actually a data processing algorithm. Despite the typical description of a filter as a "black box" containing electrical networks, the fact is that in most practical applications, the "filter" is just a computer program in a central processor. As such, it inherently incorporates discrete time measurement samples rather than continuous time inputs.

Figure 1 depicts a typical situation in which a Kalman filter could be used advantageously. A system of some sort is driven by some known controls, and measuring devices provide the value of certain pertinent quantities. Knowledge of these system inputs and outputs is all that is explicitly available from the physical system for estimation purposes. The need for a filter now becomes apparent. Often the variables of interest, some finite number of quantities to describe the "state" of the system, cannot be measured directly, and some means of inferring these values from the available data must be generated. This inference is complicated by the facts that the system is typically driven by inputs other than our own known controls and that the relationships among the various "state" variables and measured outputs are known only with some degree of uncertainty. Furthermore, any measurement will be corrupted to some degree by noise, biases, and device inaccuracies, and so a means of extracting valuable information from a noisy signal must be provided as well. There may also be a number of different measuring devices, each with its own particular dynamics and error characteristics, that provide some information about a particular variable, and it would be desirable to combine their outputs in a systematic and optimal manner. A Kalman filter combines all available measurement data, plus prior knowledge about the system and measuring devices, to produce an estimate of the desired variables in such a manner that the error is minimized statistically. In other words, if we were to run a number of candidate filters many times for the same application, then the average results of the Kalman filter would be better than the average results of any other.

2 Kalman Filter Formulation

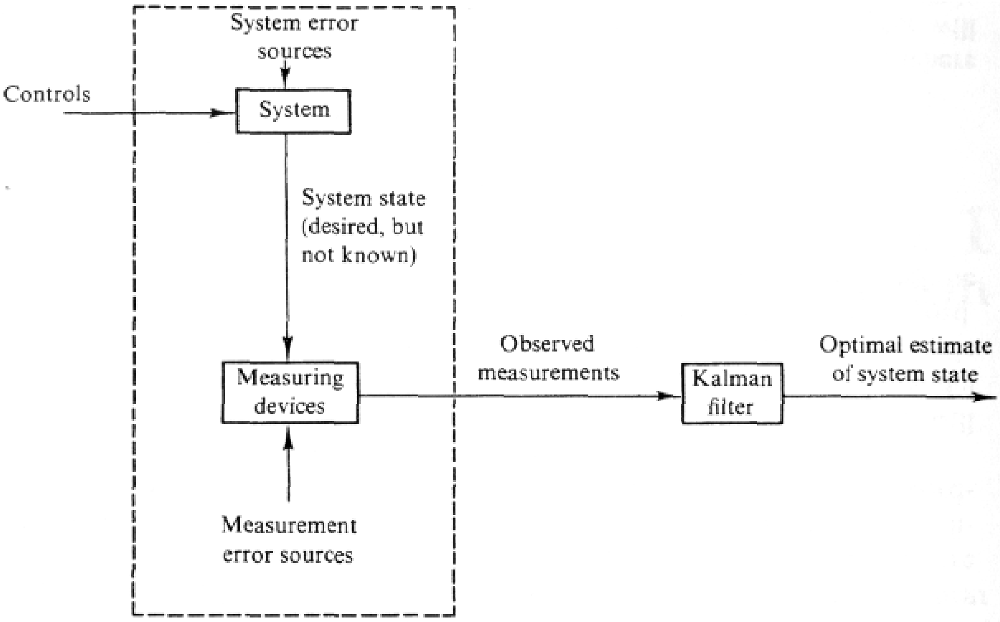

Figure 1: Typical Kalman filter application

Consider the stochastic linear discrete-time dynamic system

$$
\begin{aligned}
x_{k+1} &= A_k x_k + B_k u_k + D_k w_k & (1)\\
y_k &= C_k x_k + E_k w_k & (2)
\end{aligned}
$$

where $A_k \in R^{n_{k+1}\times n_k}, B_k \in R^{n_{k+1}\times m_k}, C_k \in R^{p_k \times n_k}, D_k \in R^{n_{k+1}\times q_k}, E_k \in R^{p_k\times q_k}$ are known matrices. Note that A_k need not be square and may have time-varying size. Assume that, for all $k \geq 0$, the input $u_k \in R^{m_k}$ is known and the output $y_k \in R^{p_k}$ is measured. The noise $w_k \in R^{q_k}$ is assumed to be white, Gaussian, zero mean, and with unit covariance $\varepsilon\left[w_k w_k^T\right] = I_{q_k}$, where $\varepsilon\left[\cdot\right]$ denotes expected value. $D_k w_k$ represents process noise, while $E_k w_k$ is the output noise. If D_k and E_k are chosen such that $D_k E_k^T = 0_{n_{k+1}\times p_k}$,then the process noise and the output noise are uncorrelated.

For the system (1), (2), we consider a one-step predictor of the form

$$\widehat{x}_{k+1} = A_k \widehat{x}_k + B_k u_k + L_k(y_k - \widehat{y}_k), \tag{3}$$

where

$$\widehat{y}_k = C_k \widehat{x}_k \tag{4}$$

and the predictor gain $L_k \in R^{n_{k+1}\times p_k}$ minimizes the loss function

$$J_k(L_k) \triangleq \varepsilon\left[(x_{k+1} - \widehat{x}_{k+1})^T \times W_k(x_{k+1} - \widehat{x}_{k+1})\right],$$

where $W_k \in R^{n_{k+1}\times n_{k+1}}$ is assumed to be positive definite.

Next, define the prediction error e_k and the innovation ν_k by

$$
\begin{aligned}
e_k &\triangleq x_k - \widehat{x}_k, & (5)\\
\nu_k &\triangleq y_k - \widehat{y}_k,
\end{aligned}
$$

and the prediction error covariance P_k^{xx} and the innovation covariance P_k^{yy} by

$$\begin{aligned} P_k^{xx} &\triangleq \varepsilon\left[e_k e_k^T\right], \\ P_k^{yy} &\triangleq \varepsilon\left[\nu_k \nu_k^T\right]. \end{aligned}$$

It follows from (1)-(4) that

$$\begin{aligned} e_{k+1} &= (A_k - L_k C_k)e_k + (D_k - L_k E_k)w_k, && (6) \\ \nu_k &= C_k e_k + E_k w_k, && (7) \end{aligned}$$

the following lemma is needed.

Lemma 1

The prediction error given by (5) satisfies

$$\varepsilon\left[e_k w_k^T\right] = 0_{n_k \times q_k}. \quad (8)$$

Proof

From (6) we have

$$e_k w_k^T = (A_k - L_k C_k)e_{k-1} w_k^T + (D_k - L_k E_k)w_{k-1} w_k^T.$$

As e_{k-1} and w_k, w_{k-1} and w_k are uncorrelated. Then

$$\begin{aligned} \varepsilon\left[e_k w_k^T\right] &= (A_k - L_k C_k)\varepsilon\left[e_{k-1} w_k^T\right] + (D_k - L_k E_k)\varepsilon\left[w_{k-1} w_k^T\right] \\ &= 0 \end{aligned}$$

as $\varepsilon\left[e_{k-1} w_k^T\right]$ and $\varepsilon\left[w_{k-1} w_k^T\right]$ both equal to zero.

Proposition 1

For the predictor (3), (4), the prediction error covariance P_k^{xx} is updated by

$$P_{k+1}^{xx} = A_k P_k^{xx} A_k^T + Q_k + L_k P_k^{yy} L_k^T - P_k^{xy} L_k^T - L_k P_k^{xy}, \quad (9)$$

where

$$\begin{aligned} P_k^{yy} &= C_k P_k^{xx} C_k^T + R_k, && (10) \\ P_k^{xy} &\triangleq A_k P_k^{xx} C_k^T + S_k, && (11) \end{aligned}$$

and

$$Q \triangleq D_k D_k^T, R_k \triangleq E_k E_k^T, S_k \triangleq D_k E_k^T. \quad (12)$$

Proof

By definition

$$\begin{aligned} P_k^{yy} &= \varepsilon\left[\nu_k \nu_k^T\right] \\ &= \varepsilon\left[(C_k e_k + E_k w_k)(C_k e_k + E_k w_k)^T\right] \\ &= \varepsilon\left[C_k e_k e_k^T C_k^T + C_k e_k w_k^T E_k^T + E_k w_k e_k^T {C_k}^T + E_k w_k w_k^T E_k\right] \\ &= C_k \varepsilon\left[e_k e_k^T\right] C_k^T + C_k \varepsilon\left[e_k w_k^T\right] E_k^T + E_k \varepsilon\left[w_k e_k^T\right] {C_k}^T + E_k \varepsilon\left[w_k w_k^T\right] E_k \end{aligned}$$

as $\varepsilon\left[e_k w_k^T\right]=\varepsilon\left[w_k e_k^T\right]=0, \varepsilon\left[w_k w_k^T\right]=1$, we have

$$\begin{aligned} P_k^{yy} &= C_k P_k^{xx} C_k^T + E_k E_k^T \\ &= C_k P_k^{xx} C_k^T + R_k. \end{aligned}$$

Meanwhile,

$$\begin{aligned} P_{k+1}^{xx} &= \varepsilon\left[e_{k+1} e_{k+1}^T\right] \\ &= \varepsilon\left[((A_k - L_k C_k) e_k + (D_k - L_k E_k) w_k)(e_k^T (A_k^T - C_k^T L_k^T) + w_k^T (D_k^T - E_k^T L_k^T))\right] \\ &= \varepsilon[(A_k - L_k C_k) e_k e_k^T (A_k^T - C_k^T L_k^T) + (A_k - L_k C_k) e_k w_k^T (D_k^T - E_k^T L_k^T) + \\ &\quad (D_k - L_k E_k) w_k e_k^T (A_k^T - C_k^T L_k^T) + (D_k - L_k E_k) w_k w_k^T (D_k^T - E_k^T L_k^T)] \\ &= (A_k - L_k C_k) P_k^{xx} (A_k^T - C_k^T L_k^T) + \varepsilon\left[D_k D_k^T\right] + \varepsilon\left[L_k E_k E_k^T L_k^T\right] - \varepsilon\left[D_k E_k^T L_k^T + L_k E_k D_k^T\right] \\ &= (A_k - L_k C_k) P_k^{xx} (A_k^T - C_k^T L_k^T) + Q_k + L_k R_k L_k^T - S_k L_k^T - L_k S_k^T \\ &= A_k P_k^{xx} A_k^T + L_k C_k P_k^{xx} C_k^T L_k^T - L_k C_k P_k^{xx} A_k^T - A_k P_k^{xx} C_k^T L_k^T + Q_k + L_k R_k L_k^T - S_k L_k^T - L_k S_k^T \\ &= A_k P_k^{xx} A_k^T + Q_k + L_k (C_k P_k^{xx} C_k^T + R_k) L_k^T - (A_k P_k^{xx} C_k^T + S_k) L_k^T - L_k (C_k P_k^{xx} A_k^T + S_k^T) \\ &= A_k P_k^{xx} A_k^T + Q_k + L_k P_k^{yy} L_k^T - P_k^{xy} L_k^T - L_k P_k^{xy} \\ &= (13). \end{aligned}$$

Also

$$\begin{aligned} J_k(L_k) &= \varepsilon\left[e_{k+1}^T W_k e_{k+1}\right] \\ &= tr(P_{k+1}^{xx} W_k). \end{aligned} \tag{13}$$

Proposition 2

Assume that, for all $k \geq 1, P_k^{yy}$ given by (10) is positive definite. If L_k minimizes (13), then L_k is given by

$$L_k = P_k^{xy} (P_k^{yy})^{-1}, \tag{14}$$

where the error covariance P_k^{xx} in (9) is updated by the Riccati equation

$$P_{k+1}^{xx} = A_k P_k^{xx} A_k^T + Q_k - L_k P_k^{yy} L_k^T. \tag{15}$$

Proof

$$\begin{aligned} \frac{\partial J_k}{\partial L_k} &= \frac{\partial tr(P_{k+1}^{xx} W_k)}{\partial L_k} \\ &= \frac{\partial tr(A_k P_k^{xx} A_k^T W_k)}{\partial L_k} + \frac{\partial tr(Q_k W_k)}{\partial L_k} + \frac{\partial tr(L_k P_k^{yy} L_k^T W_k)}{\partial L_k} \\ &\quad - \frac{\partial tr(P_k^{xy} L_k^T W_k)}{\partial L_k} - \frac{\partial tr(L_k P_k^{xy} W_k)}{\partial L_k} \\ &= \frac{\partial tr(L_k P_k^{yy} L_k^T W_k)}{\partial L_k} - \frac{\partial tr(P_k^{xy} L_k^T W_k)}{\partial L_k} - \frac{\partial tr(L_k P_k^{xy} W_k)}{\partial L_k} \\ &= 2 P_k^{yy} L_k^T W_k - 2 P_k^{xy} W_k. \end{aligned}$$

Hence, $\frac{\partial J_k}{\partial L_k} = 0 \; iff$

$$L_k = P_k^{xy}(P_k^{yy})^{-1}.$$

If (14) is substituted into (9),

$$\begin{aligned} P_{k+1}^{xx} &= A_k P_k^{xx} A_k^T + Q_k + P_k^{xy}(P_k^{yy})^{-1} P_k^{yy} L_k^T - P_k^{xy} L_k^T - L_k P_k^{yy} (P_k^{yy})^{-1} P_k^{xy} \\ &= A_k P_k^{xx} A_k^T + Q_k + P_k^{xy} L_k^T - P_k^{xy} L_k^T - L_k P_k^{yy} L_k^T \\ &= A_k P_k^{xx} A_k^T + Q_k - L_k P_k^{yy} L_k^T, \end{aligned}$$

which proves (15).

Proposition 3

L_k given by (14) is the unique global minimizer of $J_k(L_k)$ given by (13).

References

1. http://en.wikipedia.org/wiki/Kalman_filter
2. Teixeira, B.O.S., *Kalman Quickie*, IEEE Control Systems Magazine, Vol. 30, No. 3, pp. 17-18, June 2010