ELEC9733 Real Time Computing & Control
Linear Quadratic Control

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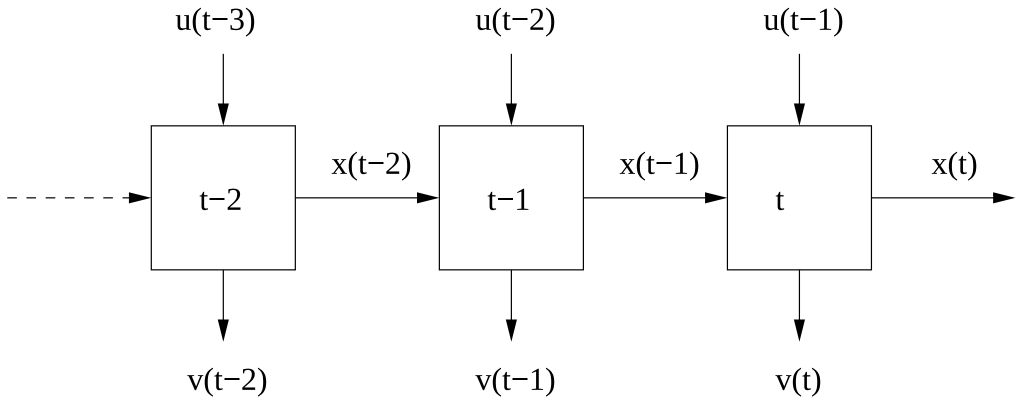
A Linear Quadratic Controller minimises a quadratic cost function:

$$C = \sum_{t=1}^{N} X_t^\top P X_t + U_{t-1}^\top Q U_{t-1} = \sum_{t=1}^{N} v_t$$

This can be represented as a multi-stage optimisation problem, and solved by dynamic programming.



The state updating is given by

$$x_{t+1} = Ax_t + Bu_t$$

We can now compute the control signal sequence that will optimise the cost using backward iterations and dynamic programming.

1 Stage Optimisation

$$r_1 = x_t^\top P x_t + u_{t-1}^\top Q u_{t-1} = (Ax_{t-1} + Bu_{t-1})^\top P(Ax_{t-1} + Bu_{t-1}) + u_{t-1}^\top Q u_{t-1}$$

Optimising with respect to u_{t-1}

$$
\begin{aligned}
\frac{dr_1}{du_{t-1}} &= 2B^\top P_1(Ax_{t-1} + Bu_{t-1}) + 2Qu_{t-1} = 0 \\
u_{t-1} &= -(Q + B^\top P_1 B)^{-1} B^\top P_1 A x_{t-1} = -K_{t-1} x_{t-1} \\
r_1^* &= (Ax_{t-1} - BK_{t-1}x_{t-1})^\top P_1 (Ax_{t-1} - BK_{t-1}x_{t-1}) + (K_{t-1}x_{t-1})^\top Q (K_{t-1}x_{t-1})
\end{aligned}
$$

r_1^* is the optimised cost.

2 Stage Optimisation

The two-stage cost is the optimal one-stage cost plus the added cost of control for the preceding stage, i.e.

$$
\begin{aligned}
r_2 &= r_1^* + x_{t-1}^\top P x_{t-1} + u_{t-2}^\top Q u_{t-2} \\
&= x_{t-1}^\top \left[P + (A - BK_{t-1})^\top P_1 (A - BK_{t-1}) + K_{t-1}^\top Q K_{t-1} \right] x_{t-1} + u_{t-2}^\top Q u_{t-2}
\end{aligned}
$$

This has exactly the same form as the cost for the one stage optimisation if we make the following replacement:

$$
P_2 = P + (A - BK_{t-1})^\top P_1 (A - BK_{t-1}) + K_{t-1}^\top Q K_{t-1}
$$

In this case

$$
u_{t-2} = -(Q + B^\top P_2 B)^{-1} B^\top P_2 A x_{t-2} = -K_{t-2} x_{t-2}
$$

General Solution

Interating over more stages will obviously produce similar results, and we can see that the general solution can be given as:

$$
\begin{aligned}
P_i &= P + (A - BK_{i-1})^\top P_{i-1} (A - BK_{i-1}) + K_{i-1}^\top Q K_{i-1} \\
K_i &= (Q + B^\top P_i B)^{-1} B^\top P_i A \\
u_i &= -K_i x_i
\end{aligned}
$$

Combining the first two equations yields what is known as the *Riccati Equation* which needs to be iterated to steady state to derive LQ control.